TEST OF THE SLUTSKY CONDITIONS

By

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1. Introduction

In practice, model builders usually proceed along one of three paths to build a link between the general demand functions and statistical demand models. The first approach assumes an explicit form for a utility function from which the corresponding demand function is derived. Such an approach is the one taken in the monographs by Houthakker and Taylor (1970) and Betancourt (1971 : 283-292).

The second approach involves first the selection of a certain "theoretically plausible" functional form for the demand curves that does not violate a limited set of theoretical properties of demand functions. The structure of the choice space can then be established from this specific pre-chosen demand relation (1).

The third approach also requires the choice of the "theoretically plausible" demand functions, but here the test of the utility maximization theory is implemented in a different manner. In this instance, it is of some interest to determine the theoretically restricted and unrestricted estimates of the parameters from sample data and test the compatibility of the demand theory with the sample information. It is expected that the empirical application of the Slutsky conditions - - the Engel aggregation, homogeneity,

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(1) See Wold and Jureen (1953 : 106-108), for Törnquist's system of demand functions; Leser (1963 : 694-703) (various forms of Engel curves that satisfy the Engel aggregation condition are introduced); Prais (1953 : 87-103); Liviatian (1964: 34) (cumulative log-normal function for aggregates); Houthakker and Taylor (1970 : 8) (forms for time series data : linear, double-log, semi-log, and inverse-log functions).

symmetry and negativity - - derived from consumer choice theory, provides the advantages of greater statistical efficiency in estimating demand parameters, and assurance that the parameters will fall within the framework of a consistent and soundly based theory [1]. In these types of studies, restricted simultaneous estimation is usually implemented by the use of Zellner's version of Aitken's generalized least squares procedure and restrictions imposed through the constraints in the objective function (Zellner, 1962 : 348-368).

One of the conclusive models in the final group is Court's New Zealand meat model (1967 : 424-444); the other is Barten's test of the Slutsky conditions on the parameters of the empirical demand functions for the four aggregates - - food, pleasure goods, durables and the remainder - - with the Netherlands data (1967 : 77-84). Both studies concluded that utility maximization holds in reality. On the other hand, Byron's application (1970 : 816-830) to Barten's sixteen-sector Dutch expenditure data rejects the null hypothesis that Engel aggregation, homogeneity, symmetry and negativity restrictions are compatible with the sample information.

To summarize the preceding survey, we note that existing empirical tests in the literature, using primarily the data of advanced countries, do not show consistent evidence. More empirical support in the application of the general slutsky conditions is required.

Moreover, in practice, econometricians trying to estimate a system of m demand equations confront with the estimation problem of m^2 price effects and m income effects. Since the number of parameters to be estimated becomes extremely large even for a small system of demand equations, econometricians try to use the consequences of the classical consumer's choice theory to reduce the parameters to be estimated. The Slutsky conditions, obtained from the rational individual consumer's behavior and assumed to be valid for any well behaved general utility function, have been effectively used in practice for market demand functions neglecting the aggregation biases, and assuming that the behavior of the community can be represented by the behavior of a **representative** consumer (Stone, 1954 and 1965 : 271-290; Leser, 1961 : 24-30). Application of these ordinal restrictions reduces the number of parameters to be

(1) See Theil (1961 : 829), for "... considerable gains in the efficiency of point estimates could be achieved with the use of even incorrect prior information related to the parameters of a system of demand equations..."

estimated from m(m+I) to 1/2 m(m+I) - 1. Hence, it becomes very important to see the empirical validity of the Slutsky conditions [1].

The object of the demand model formulated in this paper is to test the consistency of the prior information implied by the consumer's choice theory, and the sample information obtained from the developing country in case of broad aggregates. This test is implemented by a restricted estimation procedure, using Greece as a sample. The parameters of the demand equations for three broad aggregates, namely, necessities, semi-luxuries and luxuries, are estimated by imposing the Slutsky conditions by means of the constraints within a simultaneous estimation model assuming a non-additive utility function.

The paper is in five sections. Following the introduction, the Second Section gives a brief summary of the empirical restrictions derived from consumer's choice theory, while Section III provides information about the data used in the sample. Section IV describes the estimation procedure applied in the model, and Section V presents the findings and conclusions.

2. The Slutsky Conditions

The classical consumer's choice theory, its assumptions and conclusions are well known. The derivation of both general demand functions and their properties need not be repeated in detail here[2]. Only a brief review of the invariant notions of the fundemental theorems, called "Slutsky Conditions" in this paper, will be given below [3].

(1) If the system of demand equations contains a fairly large number of demand equations, the application of even these ordinal restrictions on m+I demand equations, seems unsatisfactory for handling the data and computation difficulties of a fully interdependent estimation model assuming a non-additive utility function. Additional constraints on the utility function are needed. These constraints have been provided by using prior notions about the interrelation of commodities with respect to the satisfaction of wants, such as assumptions of separability and want independence. For example, Frisch's and Barten's type of want independence assumption reduces the number of parameters to be estimated to something in the order of 1+2m, whereas want neutrality assumptions reduce to the order of 2m (Frisch, 1959 : 144-145; Barten, 1968 ; 241: Pearce, 1964 : 185).

(2) See Slutsky (1962 : 27-56), for the derivation of both general demand functions and their theoretical properties; see also Samuelson (1948 : 113-115), for the empirical implications of the utility analysis.

(3) Restrictions imposed upon the empirical demand functions are referred to

Using the notation,

(1) $$e_{ij} = \frac{\partial q_i}{\partial p_j} \cdot \frac{p_j}{q_i},$$

the jth price elasticity of the i th commodity,

(2) $$E_i = \frac{\partial q_i}{\partial y} \cdot \frac{y}{q_i},$$

(3) $$w_i = \frac{p_i q_i}{y},$$

the proportion of income spent on the ith commodity, called the budget proportion or value share, the Slutsky conditions stated in terms of elasticities are :

(4) $$\sum_{i=1}^{m} w_i E_i = 1,$$

(5) $$\sum_{j=1}^{m} e_{ij} + E_i = 0 \qquad (i = 1, \ldots, m),$$

or

$$\sum_{j=1}^{m} e_{ij} = -E_i \qquad (i = 1, \ldots, m).$$

The first equation states the Engle aggregation condition or the adding-up property and means that the sum of the income elasticities weighted by their respective expenditure budget proportions should be equal to unity. In another terms, it implies that an increase in total expenditure is completely allocated to all commodities in the budget. The homogeneity condition, stated in equation (5), implies the absence of money illusion for individual consumers and means that for the individual demand of any commodity the sum of direct and cross price elasticities is equal to the income elasticity in absolute value but with a reverse sign.

The symmetry of the Slutsky income compensated substitution effects in terms of elasticities may be written either in the form,

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under different headings in the literature, for example, "meaningful theorems" by Samuelson (1948 : 107); "verifable theorems" by Stone and Rowe (1954-1963); and "Slutsky conditions" by Barten (1967 : 77).

$$(6) \qquad \frac{e_{ij}}{w_j} + E_i = \frac{e_{ji}}{w_i} + E_j, \quad (i,j = 1, \ldots, m),$$

or in terms of the Hicks-Allen elasticities of substitution.

$$(7) \qquad \sigma_{ij} = e_{ij} + w_j E_i = e_{ji} + w_i E_j = \sigma_{ji},$$

where the σ_{ij} terms denote net cross price elasticity of substitution, and the e_{ij} terms indicate the gross price elasticities of substitution. The Slutsky symmetry condition means that an incremenal change of the j th good in the case of a compensated variation of price p_i is equal to the incremental change of the i th good in the case of a compensated variation of the price p_j. That is to say, only the net cross price elasticities of the two goods are equal to each other.

The negativity condition expressed in terms of direct net substitution elasticity, $\sigma_{ii} < 0$, provides the justification for the intuitive notion of the negative relationship between quantity and prices. As is shown first by Slutsky, although it is not possible to deduce an unequivocal sign condition for income elasticity, and, in turn, for gross price elasticity, it is certain that the direct net substitution effect will have a negative sign[1] As a consequence, only in the case of **normal** commodities, when the price goes up a decrease should be expected in the quantity bought.

When all of the net substitution elasticities are written as the element of a matrix :

$$(8) \qquad (\sigma_{ij}) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2m} \\ \vdots & \vdots & & \vdots \\ \sigma_{m1} & \sigma_{m2} & \ldots & \sigma_{mm} \end{bmatrix}$$

This is called the substitution matrix. Its properties provide the negatitivity, homogeneity and symmetry restrictions all together. That is to say, it has to be symmetrical and in negative semi-definite quadratic form. The determinants,

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(1) That is : $k_{ii} = \lambda \frac{D_{ii}}{D} = \frac{\partial q_i}{\partial p_i} + q_i \frac{\partial q_i}{\partial y} < 0$.

$$(9) \qquad |\sigma_{ii}| \,, \quad \begin{vmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ji} & \sigma_{jj} \end{vmatrix} \,, \quad \begin{vmatrix} \sigma_{ii} & \sigma_{ij} & \sigma_{ik} \\ \sigma_{ji} & \sigma_{jj} & \sigma_{jk} \\ \sigma_{ki} & \sigma_{kj} & \sigma_{kk} \end{vmatrix} \quad ,\text{etc.},$$

should be alternately negative and positive, except for the determinant of (σ_{ij}), which is zero because of the homogeneity condition insuring that utility is maximized.

The Slutsky conditions expressed in elasticity form have been applied to the elasticity parameters of log-linear demand functions. Demand functions with constant elasticity are considered approximately a plausible form for broad aggregates over a certain range of income. The reasons for the choice of this form for the demand function in this section are low degrees of freedom, and easiness in both estimation and interpretation. These demand functions for a system of m broad aggregates, treated as related commodities, can be written as :

$$(10) \qquad \log q_i = e_{io} + \sum_{j=1}^{m} e_{ij} \log p_j + E_i \log y + u_i \qquad (i = 1, \ldots, m)$$

where q_i is the per capita quantity demanded of the composite i, p_j is the price of composite j, y is total per capita expenditure, and u_i is the disturbance term normally distributed with zero mean and constant variance.

More specifically, the system of three log-linear demand equations used in the sample, which represents the demand for total private consumption expenditures consisting of necessities, semi-luxuries and luxuries as sub-groups, can be stated as :

(11)

$$\log q_1 = e_{10} + e_{11} \log p_1 + e_{12} \log p_2 + e_{13} \log p_3 + E_1 \log y + u_1$$
$$\log q_2 = e_{20} + e_{21} \log p_1 + e_{22} \log p_2 + e_{23} \log p_3 + E_2 \log y + u_2$$
$$\log q_3 = e_{30} + e_{31} \log p_1 + e_{32} \log p_2 + e_{33} \log p_3 + E_3 \log y + u_3$$

The restrictions imposed on this system of demand equations can also be rewritten as follows :

$$w_1E_1 + w_2E_2 + w_3E_3 = 1 \tag{12}$$

$$\begin{aligned} e_{11} + e_{12} + e_{13} &= -E_1 \\ e_{21} + e_{22} + e_{23} &= -E_2 \\ e_{31} + e_{32} + e_{33} &= -E_3 , \end{aligned} \tag{13}$$

$$\begin{aligned} \sigma_{12} &= e_{12} + w_2E_1 = e_{21} + w_1E_2 = \sigma_{21} \\ \sigma_{23} &= e_{23} + w_3E_2 = e_{32} + w_2E_3 = \sigma_{32} \\ \sigma_{31} &= e_{31} + w_1E_3 = e_{13} + w_3E_1 = \sigma_{13} \end{aligned} \tag{14}$$

Equations (12), (13) and (14) represent respectively the restrictions of the Engel Aggregation, homogeneity and symmetry (1).

3. The Data

The data used in the application of the restricted demand model is based on household consumption surveys of Greece. Time series of both current and constant annual private consumption expenditures for fifteen commodity groups, between the years of 1953 and 1967, were obtained from **U.N. Yearbook of National Accounts Statistics** (1966 and 1969, Tables 7a and 7b), and mid-year population observations were taken from **U.N. Demographic Yearbook** (1970, Table 4).

To handle the data, first fifteen expenditure groups were aggregated into seven broad commodity classes: food, pleasure goods, clothing, rent, durables, services, and transportation, and are treated as elementary commodities. From the expenditure elasticities of these seven demand equations, the commodity groups were classified into three categories: luxuries, consisting of clothing, transportation and durables; semi-luxuries, consisting of pleasure goods, rent and services; and necessities, comprised of food only. The components of both aggregates are presented in Table 1.

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(1) As is frequently seen in literature, it is assumed here that problems related to aggregation can be ignored in the extention of classical consumer's theory from an individual to a market.

TABLE 1
BROAD AGGREGATES AND COMPONENTS

Broad Aggregates	Aggregates	Components
Necessities	Food	Food
Semi-luxuries	Pleasure Goods	Beverages Tobacco
	Rent	Rent and water charges Fuel and light Household operation
	Services	Personal care and health expenses Recreation and entertain-ment Miscellaneous services (financial, education, research and other)
Luxuries	Clothing	Clothing Other personal effects Footwear
	Durables	Furniture Furnishings Household equipment
	Transportation	Personal transport equip-ment Operation of personal transport equipment Purchased transport Communication

Using per capita data, the logarithms of the total expenditure observations were calculated from the current series, while the logarithms of the dependent variables were calculated from the constant series (1). The logarithms of the implicit price deflators were treated as price variables. Implicit price deflators were found by dividing the expenditure at current prices on the i th composite into the expenditure at constant prices on the same composite. The mean value of the budget proportions that were imposed on the parameters of the system were calculated from current time series expenditure figures.

4. The Estimation procedure

This section describes the restricted estimation procedure of the demand model which is set up to test the empirical implications of utility maximization. The problem is to estimate a system of log-linear demand equations, (11), subject to parametric restrictions.

The observations used in estimating the price and income coefficients of the three demand equations, (11), were classified as cross section units over the commodity aggregates, and had the dimentions of both time and cross section. In combining the cross-section units and time series, the approach used in this paper aims to increase the asymptotic efficiency (Gooldberger, 1964 : 357) of the estimates by taking into account the possible correlations that might exist among the disturbance terms (2).

The estimation procedure chosen is a restricted and maximum likelihood version of Zellner's iterative Aitken estimator(3). In this

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(1) In this paper total expenditure is used in preference to income since the latter is more likely to include transitory and unexpected elements. Accordingly, terms of income elasticity and expenditure elasticity are used as equivalent.

(2) In most cases, the correlations that are assumed to exist among the disturbances are taken into account either only in the time direction autoregressive schemes) or only in the cross-section units (heteroscedasticity). In some cases, however, the disturbance terms are assumed to be composed of three independent parts. One is associated only with time, and another only with the cross-section units, while the third is a combination of the interactions between both dimensions. This approach uses either known variances of the error components or unknown variances and an iterative estimation procedure. See Wallace and Huassain (1969 : 55-72).

(3) See Zellner (1962 : 348-354), for the unconstrained Zellner version of Aitken's generalized least squares; see Court (1967 : 424-444), for a constrained ML version of Zellner's model. see Byron (1970 : 817-882) for a constrained GLS version of Zellner's model.

procedure, demand equations are set up in the manner of Zellner's application of Aitken's generalized least squares to a system of equations, and then restrictions are imposed by Lagrange multipliers. Coefficients of the system are estimated by a step-wise method [1]. A statistical test which rejects restrictions used in the estimation procedure also rejects the Slutsky conditions.

Assuming a multivariate normal linear regression model [2] the linear functional form of the i th demand equation can be written as :

(15) $y_{ti} = \beta_{1i} X_{t1} + \ldots + \beta_{Ki} X_{tK} + u_{ti}$,

where $(i = 1, \ldots, m)$, $(k = 1, \ldots, K)$, $(t = 1, \ldots, T)$.

The m relations in matrix notation can be defined as follows : the T x m dependent variable observation matrix Y is

(16) $Y = (y_1 \ldots y_i \ldots y_m)$,

where y_i is a T x 1 column vector; the T x K independent variable observation matrix X is

(17) $X = (X_1 \ldots X_k \ldots X_K)$

in which X_k denotes a T x 1 column vector; the K x m coefficient matrix B is

(18) $B = (B_1 \ldots B_i \ldots B_m)$,

where B_i denotes a K x 1 column vector; and the T x m residual matrix U is

(19) $U = (u_1 \ldots u_i \ldots u_m)$,

where u_i shows a T x 1 column vector.

The mT equations (15) can then be written compactly as :

(20) $Y = XB + U$,

in which each column refers to one of the m relations. The stochastic specifications of each i th equation are :

(21) $y_i = XB_i + u_i \qquad (i = 1, \ldots, m)$,

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(1) The step-wise maximization procedure of Koopmans and Hood is usually Used in these estimation methods. See Hood and Koopmans (1953 : 143-162).

(2) See Goldberger (1964 : 207-211), for the multivariate classical linear regression model.

(22) $Eu_i = 0$ $(i = 1, \dots, m)$,

(23) $Eu_i u'_i = w_{ii} I$ $(i = 1, \dots, m)$,

(24) X is a T x K matrix which is fixed in repeated samples, and

(25) Rank of $X = K \leq T$.

Allowing the disturbances in different demand equations to be correlated with each other for the same t :

(26) $Eu_i u'_{i'} = w_{ii'} I$ $(i, i' = 1, \dots, m : i \neq i')$.

In order to define the specifications, (21) through (26), we can define the T x m disturbance matrix as

(27) $\varepsilon = (\varepsilon_1 \dots \varepsilon_i \dots \varepsilon_m)$,

in which ε_i is a T x l column vector, and the mxm disturbance contemporaneous covariance matrix is :

(28) $$\Omega = E\varepsilon_i(t)\ \varepsilon'_i(t) = \begin{bmatrix} w_{11} & \dots & w_{1m} \\ \vdots & w_{ii} & \vdots \\ w_{m1} & \dots & w_{mm} \end{bmatrix}$$

Hence, the stochastic specifications of the multivariate model are :

(29) $Y = XB + \varepsilon$,

(30) $E\varepsilon = 0$,

(31) $$E\,\varepsilon_i(t)\,\varepsilon_i'(t) = \begin{cases} \Omega & \text{if } t = t' \\ 0 & \text{if } t \neq t' \end{cases},$$

(32) X is a T x K matrix which is fixed in repeated samples, and

(33) Rank of $X = K \leqq T$.

The m demand equations, (21), which were specified as a full system under the multivariate classical normal linear regression model, (29), can be written as :[1]

(34) $$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} X & \Phi & \dots & \Phi \\ \Phi & X & \dots & \Phi \\ \vdots & \vdots & & \vdots \\ \Phi & \Phi & \dots & X \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

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(1) See Goldberger (1964 : 246 and 262), for this form and the explanation of the fact that if $X_1 = \dots = X_m$ are equal, then OLS estimators are the

where Φ denotes the null matrix of the order T x k. Alternatively, this can be written in the following form :

(35) $\quad y = (I_m \otimes X)\, b + u,$

where y and u are mT x 1 vectors, b is a mk x 1 vector, I_m is the unit matrix of order m, and the symbol $\otimes$ represents the Kronecker product operation.

The set of linear restrictions on the elements of b can be written as :

(36) $\quad Rb = \emptyset,$

where R is a J x mk restriction matrix whose elements are the appropriate coefficients, $\pm\bar{w}_i$, ± 1 or 0, of the restrictions introduced in equations (12), (13) and (14); J is the number of restrictions on the whole system; and $\emptyset$ is the J x l null vector. Ignoring the constant terms, this equation can be shown in the way it was used in the application of the model :

(37)
$$\left[\begin{array}{cccc|cccc|cccc} 0 & 0 & 0 & \bar{w}_1 & 0 & 0 & 0 & \bar{w}_2 & 0 & 0 & 0 & \bar{w}_3 \\ \hline 0 & 1 & 0 & \bar{w}_2 & -1 & 0 & 0 & -\bar{w}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{w}_3 & 0 & -1 & 0 & -\bar{w}_2 \\ 0 & 0 & -1 & -\bar{w}_3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \bar{w}_1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \\ E_1 \\ \hline e_{21} \\ e_{22} \\ e_{23} \\ E_2 \\ \hline e_{31} \\ e_{32} \\ e_{33} \\ E_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\bar{w}_i$ indicates mean values of the budget proportions and the e vector indicates gross price and income elasticities. The first row of the restriction matrix represents the Engle aggregation (12), the

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BLUE's of the coefficients in the multivariate classical linear regression model. However, if $X_1 \neq \cdots \neq X_m$, then Zellner's two-stage Aitken estimator is more efficient than the OLS estimator. This is result of the apriori information on the coefficients of B, (29).

next three rows show the symmetry condition, and last three show the homogeneity condition.
That is :

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\begin{aligned}
&\bar{w}_1 E_1 + \bar{w}_2 E_2 + \bar{w}_3 E_3 = 1 \\
&e_{12} + \bar{w}_2 E_1 - e_{21} - \bar{w}_1 E_2 = 0 \\
&e_{23} + \bar{w}_3 E_2 - e_{32} - \bar{w}_2 E_3 = 0 \\
(38)\quad &e_{31} + \bar{w}_1 E_3 - e_{13} - \bar{w}_3 E_1 = 0 \\
&e_{11} + e_{12} + e_{13} + E_1 = 0 \\
&e_{21} + e_{22} + e_{23} + E_2 = 0 \\
&e_{31} + e_{32} + e_{33} + E_3 = 0 \quad .
\end{aligned}
$$

The basic statistical assumptions of the model are the stochastic specifications of the multivariate classical normal linear regression model introduced in equations (29) through (33). In other words, the u_{it} are a set of error components that are jointly normally distributed (over i) but serially independent (over t), and have zero means and finite variances and covariances. They are distributed independently of the independent variable, and X'X is non-singular.

Assuming u'_t represents the row vector $(u_{1t}, \dots, u_{mt})$ of (19), which is multivariate normal, the joint density of the u_{it} in each time period can be written as (Goldberger, 1964 : 356) :

$$(39) \quad f(u_{1t},\dots,u_{mt}) = (2\pi)^{-m/2} \, |\Omega|^{-1/2} \exp.\{-\tfrac{1}{2} u'(t)\, \Omega^{-1} u(t)\} \qquad (t = 1, \dots, T),$$

where Ω denotes the m x m disturbance contemporaneous covariance matrix, (28).

Then, since the disturbances are temporarily independent, the likelihood of the sample is (Anderson, 1958 : 180 and Goldberger, 1964 : 211) :

(40)

$$L = \prod_{t=1}^{T} f[u(t)] = (2\pi)^{-mT/2} |\Omega|^{-T/2} \exp\{-\tfrac{1}{2} \sum_{t=1}^{T} u'(t)\,\Omega^{-1} u(t)\}.$$

By using (35) and logarithms, we may construct the logarithm of the joint likelihood function as (Court, 1967 : 432) :

(41)

$$\log L = \sum_{t=1}^{T} \log f(u_{1t}, \dots, u_{mt})$$

$$= -\frac{Tm}{2}\log(2\pi) - \frac{T}{2}\log|\Omega| - \frac{1}{2}(y - Xb)'(\Omega^{-1}\otimes I)(y - Xb)$$

$$\approx \quad - T\log|\Omega| - u'(\Omega^{-1}\otimes I_T)u$$

$$\approx \quad \log|\Omega^{-1}\otimes I_T| - u'(\Omega^{-1}\otimes I_T)u.$$

The likelihood function can be maximized subject to the constraint that includes the restrictions, by means of the Lagrangean multipliers. That is :

(42) $\text{Max: } L(\beta,\Omega,\lambda) = \log|\Omega^{-1}\otimes I_T| - u'(\Omega^{-1}\otimes I_T)u - 2\lambda'Rb$

with respect to the elements of Ω^{-1}, b, and λ. λ denotes a J x 1 vector of lagrange multipliers where J is the number of restrictions in restriction matrix R. Then the values of b, λ, and Ω can be solved by equating the partial derivatives of the function to zero(1).

Since the maximum likelihood estimator of Ω is $\hat{\Omega} = T^{-1} U'U = \Sigma^*$, where $\Sigma$is the variance-covariance matrix of calculated residuals from the equations, we can regard Σ^* as a fixed, known matrix (*denotes an estimate for which the restrictions hold). Then, the estimates $\hat{b}$ and $\hat{\lambda}$, which are conditional on the initially assumed values for Σ, can be obtained by differentiating

(43) $L(b,\lambda) = - u'(\Sigma^{-1} \otimes I_T)u - 2\lambda'Rb$

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(1) Given the appropriateness of the assumption that the error vector is yielded by a multivariate joint normal process and is serially independent, it could be, say, that the above estimation procedure is within the FI/ML (full information maximum likelihood) estimator class. See Goldberger (1964 : 356 and 363).

with respect to b and λ and equating the first partials to zero. The restricted estimates of b^* in turn can be used to estimate the new restricted error vector u^*, and hence the restricted variance-covariance matrix Σ^* can also be calculated(1). This step-wise process is repeated until the estimates, b^*, converge.

Thus, regarding Σ as fixed and known, and substituting $y - (I_m \otimes X)\, b$ for u (mT x 1), we obtain the following equation :

(44)

$$L(b,\lambda) = - [y' - b'(I_m \otimes X')](\Sigma^{-1} \otimes I_T)[y - (I_m \otimes X)b] - 2\lambda' Rb .$$

Multiplying this out:

(45)

$$\begin{aligned} L(b,\lambda) &= - [y'(\Sigma^{-1} \otimes I_T) - b'(I_m \otimes X')(\Sigma^{-1} \otimes I_T)][y - (I_m \otimes X)b] - 2\lambda' Rb \\ &= - y'(\Sigma^{-1} \otimes I_T)y + b'(I_m \otimes X')(\Sigma^{-1} \otimes I_T)y \\ &\quad + y'(\Sigma^{-1} \otimes I_T)(I_m \otimes X)b - b'(I_m \otimes X') \\ &\quad (\Sigma^{-1} \otimes I_T)(I_m \otimes X)b - 2\lambda' Rb \\ &= - y'(\Sigma^{-1} \otimes I_T)y + 2b'(I_m \otimes X')(\Sigma^{-1} \otimes I_T)y \\ &\quad - b'(I_m \otimes X'X)(\Sigma^{-1} \otimes I_m)b - 2\lambda' Rb \\ &= - y'(\Sigma^{-1} \otimes I_T)y + 2b'(\Sigma^{-1} \otimes X')y \\ &\quad - b'(\Sigma^{-1} \otimes X'X)b - 2\lambda' Rb \quad . \end{aligned}$$

Differentiating the last equation with respect to vectors b and λ:

(46)
$$\frac{\partial L}{\partial b} = 2(\Sigma^{-1} \otimes X')y - 2(\Sigma^{-1} \otimes X'X)b - 2R'\lambda$$

$$\frac{\partial L}{\partial \lambda} = - 2Rb \;,$$

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(1) The same estimation of b can be derived without the normality assumption by minimizing the generalized residual variance. It can be shown that maximizing the likelihood function presented above is equivalent to minimizing the generalized reduced-form residual variance. See Goldberger (1964: 352-356) and Wonnacott, R.J. and Wonnacott, T.H. (1970: 392-394).

and setting this set of linear equations equal to zero:

(47) $$\frac{\partial L}{\partial b} = \frac{\partial L}{\partial \lambda} = \phi$$

or

(48) $$(\Sigma^{-1}\otimes X')y - (\Sigma^{-1}\otimes X'X)b - R'\lambda = \phi$$
$$Rb + \Phi\lambda = \phi$$

These become:

(49) $$(\Sigma^{-1}\otimes X'X)b + R'\lambda = (\Sigma^{-1}\otimes X')y$$
$$Rb + \Phi\lambda = \phi$$

If these equations are written in the matrix form,

(50) $$\begin{bmatrix} (\Sigma^{-1}\otimes X'X) & R' \\ R & \Phi \end{bmatrix} \begin{bmatrix} b^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} (\Sigma^{-1}\otimes X')y \\ \phi \end{bmatrix}$$

the solution of b* and λ becomes :

(51) $$\begin{bmatrix} b^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} (\Sigma^{-1}\otimes X'X) & R' \\ R & \Phi \end{bmatrix} \begin{bmatrix} (\Sigma^{-1}\otimes X')y \\ \phi \end{bmatrix}$$

where Φ is the J x J null matrix and Ø is the J x 1 null vector.

The estimated asymptotic sampling variances and covariances of the estimates of b* are given by the appropriate elements of the matrix,

$$(52) \quad \begin{bmatrix} \frac{\partial^2 L}{\partial b^2} & \frac{\partial^2 L}{\partial b \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial b} & \frac{\partial^2 L}{\partial \lambda^2} \end{bmatrix}^{-1}$$

Differentiating the first partials with the appropriate elements :

$$(53) \quad \begin{aligned} \frac{\partial^2 L}{\partial b^2} &= -2(\Sigma^{-1} \otimes X'X) , \\ \frac{\partial^2 L}{\partial b \partial \lambda} &= -2R' , \\ \frac{\partial^2 L}{\partial \lambda \partial b} &= -2R , \\ \frac{\partial^2 L}{\partial \lambda^2} &= \Phi , \end{aligned}$$

apart from irrelevant constants, the asymptotic variance-covariance matrix, (52), becomes equal to the first matrix on the right side of (51). Under general conditions, the resulting ML estimators, b* and λ*, are asymptotically efficient and consistent [1].

To apply this estimation method, an initial value for the variance-covariance matrix $\Sigma°$ has to be chosen. As a first approximation, the variance-covariance matrix is calculated from the unrestricted error terms obtained by using OLS. It is assumed that if the restrictions are valid, the unrestricted estimates will be fairly close to the restricted estimates. In this demand model, first, the unresricted b coefficients of the m demand equations, (11) or (21), were estimated using SELS, and an initial value, $\Sigma°$, was calculated. Then, replacing $\Sigma°$ in equation (51), the restricted values of b* and λ*

(1) See Kmenta and Gilbert (1968 : 1180-2000): and Goldberger (1964 : 362) for the discussion of the small sample properties of these estimators.

were obtained, completing the first cycle. For the first iteration, restricted error terms, u*, were used to calculate the restricted Σ^*. This iterative process was carried on until convergence was reached.

The equivalent of the restricted estimation problem is to test the significance of the difference between prior and sample information. If a hypothesis test indicates that the two types of information are incompatible, then the restricted parameter estimates lose much of their meaning.

This testing problem is a multivariate generalization of the general linear hypothesis of normal regression models. Instead of testing for the existence of specified linear relationships between the coefficients of a single demand equation, here the specified linear relationships, which are assumed to exist between the coefficients in several demand equations, are tested (Anderson, 1958 : chap. 8). However, although some approximations exist (Wall, 1967) derivation of the theoretical distributions of these estimators, especially in small size samples, still requires more work in the econometric and statistics theory (Goldberger, 1964 : 360; Wonnacott and Wonnacott, 1970 : 397; Overall and Klett, 1972 : 316) Meanwhile, in literature, asymptotic distributions are used to test the compatibility of prior and sample information (Court, 1967 : 435; Byron, 1970 : (821) ([1]).

Among several methods of testing the compatibility of the restricted and untrestriced parameters, one approach is to compare the restricted and unrestricted variance-covariance matrices. This is analogous to the F test of the linear hypothesis in univariate regression models (Anderson, 1958 : 188 and 210). First, we formed the determinantal values (generalized variances) of both the restricted and unrestricted variance-covariance matrices and then found the determinantal ratio by dividing the unrestricted matrix into the restricted one :

$$\frac{|\Sigma^{\circ}|}{|\Sigma^*|} \tag{54}$$

If the restrictions hold, the determinantral ratio will be close to 1. To test the significance of the difference from 1, the asymptotic likelihood ratio test was used by means of the statistic,

(1) Also see Byron (1970 : 819-821), for the Wald, the likelihood ratio, and Hotelling's T^2 tests which can be used equivalently.

$$(55) \qquad - H \log_e \frac{|\Sigma°|}{|\Sigma^*|}$$

which is distributed approximately as χ^2 with J degrees of freedom, where $H = T - \frac{1}{2}(m + J + 1)$ and T is the sample size, m is the number of commodity groups and J is the number of restrictions (Anderson, 1958 : Chap. 8).

In addition, substitution matrix (8) and the standard errors of the elements σ_{ij} were calculated to see if the matrix satisfied the requirements of utility maximization.

Despite doubts about the small sample properties of this type of estimator, empirical tests seem to be giving encouraging results (Goldberger, 1964: 362). In many empirical studies, it is shown that Zellner's two-stage and iterative Aitken estimators are more efficient than other estimators (Kmenta and Gilbert, 1968: 1199). It is also indicated that an increase in statistical efficiency can be gained by imposing theoretical restrictions upon demand equations. (Court, 1967 : 437 and Byron, 1970 : 829).

V. Findings and Conclusions

In both the unrestricted and restricted estimations, the constant terms were disposed of by using the deviations from the means. As indicated in equations (37) and (38), restrictions were imposed on the average values of the budget proportions of each commodity group. $\bar{w}_1 = .412$, $\bar{w}_2 = .366$, and $\bar{w}_3 = .222$ indicate respectively the fifteen-year averages of the necessity, semiluxury and luxury categories that were used in the restricted estimation procedure.

First unrestricted elasticities of the three demand equations were estimated by using SELS. Estimated error terms of each equation were used in the calculation of the initial variance-covariance matrix, $\Sigma°$. Then, applying the estimation procedure explained in Section IV, (51), the restricted elasticities and errors were produced with the second iteration. There were only very minor changes in subsequent iterations.

The estimates of unrestricted and restricted expenditure and price elasticities and the Lagrange coefficients are given in Table 2. The Durbin-Watson statistics in the case of unrestricted demand

TABLE 2

UNRESTRICTED AND RESTRICTED EXPENDITURE AND PRICE ELASTICITIES AND λ VALUES

Commodity Groups	Unrestricted Elasticities: Necessities	Semi Luxuries	Luxuries	Expenditure
Necessities	-0.383	-0.209	-0.095	0.680
	(0.206)	(0.034)	(0.068)	(0.076)
Semi-Luxuries	-0.250	-0.462	-0.477	1.060
	(0.427)	(0.070)	(0.141)	(0.158)
Luxuries	-0.596	-0.335	-0.046	1.399
	(0.626)	(0.103)	(0.207)	(0.233)

Commodity Groups	Restricted Elasticities: Necessities	Semi Luxuries	Luxuries	Expenditure
Necessities	-0.350	-0.200	-0.093	0.645
	(0.025)	(0.021)	(0.012)	(0.010)
Semi-Luxuries	-0.424	-0.311	-0.394	1.125
	(0.020)	(0.024)	(0.008)	(0.016)
Luxuries	-0.541	-0.669	-0.232	1.440
	(0.022)	(0.009)	(0.005)	(0.026)

	λ Values[a]: Symmetry	Homegeneity	Engel Aggregation
Necessities	2.635	39.310	
	(.468)	(3.046)	
Semi-Luxuries	-3.060	38.642	76.602
	(.525)	(2.876)	(8.416)
Luxuries	.963	19.959	
	(.282)	(1.546)	

[a]All entries are to be multiplied by 10^{-2}

equations are 2.99 for necessities, 1.17 for semi-luxuries, and .95 for luxuries. In the case of restricted demand equations the corresponding values are respectively 2.86, 1.04 and 1.05. At the 5 percent significance level for n = 15 and k = 4 none of these values fall below the lower bound d_L. An examination of the elasticities indicate that the restrictions are satisfied. The horizontal sum of the restricted elasticities is equal to zero for each equation, confirming that the homogeneity condition, (13), is met. When the restricted expenditure elasticities are multiplied by the given corresponding budget proportions, the vertical sum of the products also equals unity, revealing that the Engel aggregation condition, (12), is satisfied.

To check for the symmetry requirement, (14), the elements of the substitution matrix (σ_{ij}), (8), were calculated by using the restricted estimates and the given average budget proportions. Standard errors of the substitution elasticities were estimated from the estimated variance-covariance matrix of the restricted elasticities [1] The results are shown below.

$$(56) \qquad (\hat{\sigma}_{ij}) = \begin{bmatrix} -.085 & .036 & .050 \\ .036 & .100 & -.142 \\ .050 & -.142 & .088 \end{bmatrix}$$

and

$$(57) \qquad (s_{ij}) = \begin{bmatrix} .144 & .034 & .065 \\ .298 & .070 & .134 \\ .438 & .103 & .197 \end{bmatrix}$$

where (s_{ij}) denotes the matrix, the elements of which are standard errors of the corresponding terms in (σ_{ij}). Elements of the substitution matrix, (56), confirm that the final imposed restriction - - symmetry - - is also satisfied. However, the substitution matrix does not fulfill the other requirements - - negativity and sign conditions of quadratic form - - of the utility maximization theory.

The calculated value of the likelihood ratio is :

$$(58) \qquad -\mathrm{Hln} \frac{|\Sigma^{\circ}|}{|\Sigma^{*}|} = 39.368.$$

(1) See Wonnacott, R.J. and Wonnacott, T. (1970 : 246-247), for linear transformations and their distributions.

The χ^2 test of 7 degrees freedom show that this value is well above the one percent significance level of 18.475. Hence, the null hypothesis that the prior information and the sample information are compatible is rejected. For this reason, comparison of the restricted and unrestricted elasticities would not make economic sense.

However, it is interesting to observe that expenditure elasticities under the restrictions show little difference from the unrestricted estimates. Own price elasticities also do not show significant disparities except in luxuries. But in almost all cross elasticities considerable change is shown. This could be explained by the large number of restrictions placed on them. A comparison of the standard errors indicates considerable gain in the efficiency of the restricted estimates in spite of the use of incorrect prior information.

The rejection of the null hypothesis can be explained by either economic or statistical reasons. One possible explanation could be that the prior information is not correct. In other words, consumers in reality are not utility maximizers as has been accepted by the theory, and do experience money illusion in the face of rapidly changing prices. Particularly in the developing countries this might be expected.

Another explanation could be that the method used to test the hypothesis was not appropriate. The Slutsky restrictions are local conditions and valid for incremental changes in prices and income under the assumption of constant tastes. Here, the restrictions were imposed on the coefficients of the demand functions by means of average budget proportions, assuming constant elasticities for a time span of fifteen years. Also, assuming that the behavior of the community can be represented by the behavoir of a representative consumer and neglecting the aggregation biases might produce misleading results. More important, the sample size is too small for the interpretation of the use of asymptotic tests.

In sum, under the empirical evidence of the restricted demand model used in this paper, the rejection of the null hypothesis - - that prior and sample information are compatible - - makes it harder to arrive at a conclusion about the use of Slutsky conditions, particularly, in the demand estimation models of developing countries.

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ÖZET

SLUTKSY KISITLAMALARININ TESTİ

Tüketici tercihleri teorisinden türetilen nitel kısıtlamaların literatürdeki ampirik testleri tutarlı sonuçlar göstermemektedir. Bu çalışmada az gelişmiş bir ülkenin ana tüketim mallarından oluşturulan talep denklemler sistemi kullanılarak Slutsky kısıtlamalarının testi amaçlanmıştır. Kullanılan tahmin yöntemi kısıtlı, eş çözümlü ve en büyük olasılıklı olarak uygulanan Zellner'in iterasyonlu Aitken tahmin edicisidir (A restricted and ML version of Zellner's iterative Aitken estimator). Test Yunanistan'ın tüketim serilerine uygulanmıştır. Kısıtlı ve kısıtsız olarak tahmin edilen gelir ve fiyat esneklikleri arasındaki farklılıkların gözlenmesi oldukça ilginçtir.